A Conjecture of Mine

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Conjecture. *Let* $S : \mathbb{N} \to \mathbb{N}$ *be the sum of the digits of a natural number. Then* $S(n+m) \equiv S(n) + S(m) \pmod 9$ *for all natural numbers* n *and* m*.*

Proof. If $n = n_0 \cdot 10^0 + \cdots + n_k \cdot 10^k \in \mathbb{N}$ then:

$$
\begin{aligned}
n &= n_0 \cdot 10^0 + \cdots + n_k \cdot 10^k \\
&= n_0 \cdot (10^0 - 1 + 1) + \cdots + n_k \cdot (10^k - 1 + 1) \\
&= (n_0 \cdot (10^0 - 1) + \cdots + n_k \cdot (10^k - 1)) + (n_0 + \cdots + n_k) \\
&= S(n) + n_0 \cdot (10^0 - 1) + \cdots + n_k \cdot (10^k - 1)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
n &\equiv S(n) + n_0 \cdot (10^0 - 1) + \cdots + n_k \cdot (10^k - 1) \\
&\equiv S(n) + 0 + \cdots + 0 \\
&\equiv S(n) \pmod 9
\end{aligned}
$$

Similarly, if $m \in \mathbb{N}$ then $m \equiv S(m) \pmod 9$. Finally:

$$
\begin{aligned}
S(n+m) &\equiv n + m \\
&\equiv S(n) + S(m) \pmod 9
\end{aligned}
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